

## Finite-Energy Sum Rules and Their Application to $\pi N$ Charge Exchange\*

R. DOLEN,† D. HORN,‡ AND C. SCHMID§

California Institute of Technology, Pasadena, California

(Received 29 September, 1967)

We derive and discuss the finite-energy sum rules, which form consistency conditions imposed by analyticity on the Regge analysis of a scattering amplitude. Their finite form makes them particularly useful in practical applications. We discuss the various applications, emphasizing a new kind of bootstrap predicting the Regge parameters from low-energy data alone. We apply our methods to  $\pi N$  charge exchange and are able to derive many interesting features of the high-energy amplitudes at various  $t$ . In particular, we establish the existence of zeros of the amplitudes and of additional  $\rho$  poles. On the basis of the finite-energy sum rules and the analysis of the  $\pi N$  amplitudes, we present theoretical and experimental evidence that double counting is involved in the interference model, which adds direct-channel resonances to the exchanged Regge terms.

### I. INTRODUCTION

IN this paper we derive and discuss the finite-energy sum rules<sup>1,2</sup> (FESR), their implications, and their application to  $\pi N$  charge exchange (CEX). The main results were reported elsewhere.<sup>3</sup> Here we present a more detailed account of the theory and its application.

In Sec. II we derive FESR for arbitrary scattering amplitudes. They form consistency conditions that are imposed by analyticity alone. For amplitudes that decrease fast enough, they tend in the limit of infinite integration to the usual superconvergence relations. In our formulation, all Regge poles appear in the same form regardless of their  $\alpha$  value. This helps in particular to resolve difficulties which appear in the superconvergence relations due to the uncertainty of the exact location of the leading singularity. The finite form of the sum rules makes them particularly useful in practical applications. They are a good tool for the determination of Regge parameters from low-energy data. This section closely follows the unpublished Ref. 1.

In Sec. III we discuss the different ways of using FESR. They can be used either as a way of determining low-energy parameters through high-energy data, or in conjunction with the high-energy data for a better overall determination of the Regge fit. A third way is to predict the main parameters of the high-energy data (the exchanged Regge poles) from the low-energy data alone. This kind of bootstrap program is discussed in detail.

Sections IV and V give detailed accounts of the application of our method to  $\pi N$  CEX, as reported in Ref. 3. We use FESR to calculate the Regge parameters (as functions of  $t$ ) of the high-energy  $\pi N$  charge exchange (CEX) amplitudes from the low-energy  $\pi N$  data (phase shifts). Thus we carry out a new type of bootstrap: Given the low-energy data of  $\pi N$  (the  $N^*$  states), we calculate the exchanged  $\rho$  trajectories (masses) and coupling constants. We show how several interesting features of the  $N^*$  states "cause" (via FESR, i.e., via analyticity) corresponding features of the exchanged Regge poles.

Using the low-energy data and the FESR, we predict the following high-energy features of the  $A'^{(-)}$  and  $B^{(-)}$  amplitudes of  $\pi N$  CEX<sup>4</sup>:

- (1) The spin-flip amplitude  $\nu B^{(-)}$  is larger than the non-flip amplitude  $A'^{(-)}$  by an order of magnitude at  $t=0$ . This explains the near-forward peak in  $\pi N$  CEX.
- (2)  $B^{(-)}$  has a zero near  $t \approx -0.5$  BeV<sup>2</sup>. This explains the observed dip in  $\pi N$  CEX.
- (3)  $A'^{(-)}$  has a zero near  $t \approx -0.1$  BeV<sup>2</sup>.
- (4) In an effective one-pole model, we predict the  $\rho$  mass and a trajectory  $\alpha_{\text{eff}}$  which is 0.1 to 0.2 lower than the one measured at high energies.
- (5) Using high-energy fits as an additional input, we find some evidence for a second  $\rho$  trajectory, 0.4 lower than the  $\rho$ . This may be the manifestation of a cut.
- (6) Using the parameters of the additional  $\rho$  pole, we predict a polarization of the right sign and order of magnitude.
- (7) There is strong evidence for an (approximately) fixed pole in  $B^{(-)}$  at  $j=0$ .

The results (1) to (3) are "caused" (via FESR) by the following features (1') to (3') of the  $N^*$  states: (1') All prominent resonances enter with the same sign in  $B^{(-)}$ , but with alternating signs in  $A'^{(-)}$  at  $t=0$ . (2') All prominent resonances<sup>5</sup> have their first zero "simultaneously" in the narrow interval  $-0.6 < t < -0.4$

<sup>4</sup>  $A'^{(-)}$  and  $B^{(-)}$  in the notation of V. Singh, Phys. Rev. **129**, 1889 (1963) and G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, *ibid.* **106**, 1337 (1957).  $\nu_L$  is the laboratory energy of the  $\pi$ .

<sup>5</sup> Except the nucleon and the 1238, but they are strongly suppressed in the higher sum rules.

\* Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract No. AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission.

† Present address: Physics Department, University of California, Los Angeles, Calif. 90024.

‡ Present address: Physics Department, Tel Aviv University, Tel Aviv, Israel.

§ Present address: Lawrence Radiation Laboratory, University of California, Berkeley, Calif.

<sup>1</sup> D. Horn and C. Schmid, California Institute of Technology Report No. CALT-68-127, 1967 (unpublished).

<sup>2</sup> A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze, Phys. Letters **24B**, 181 (1967); K. Igi and S. Matsuda, Phys. Rev. Letters **18**, 625 (1967).

<sup>3</sup> R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters **19**, 402 (1967).

$\text{BeV}^2$  in  $B^{(-)}$ ; and (3') their first zero at  $-0.2 < t < -0.1$   $\text{BeV}^2$  in  $A'^{(-)}$ . For negative  $t$ , we find large cancellations between the Born term and the lowest resonances in  $A'^{(-)}$ . This can be useful for determining coupling constants of these resonances, thus checking symmetry predictions.

Section VI presents a criticism of the interference model<sup>6</sup> based on FESR. We give theoretical reasons, model examples, and experimental evidence for double counting in the interference model, which represents the amplitude at intermediate energies as a sum of Regge poles plus direct-channel resonances.

## II. DERIVATION OF THE FINITE-ENERGY SUM RULES

Finite-energy sum rules are consistency conditions imposed by analyticity on functions that can be expanded at high energies ( $\nu \geq N$ ) as a sum of Regge poles. The contribution of an individual Regge term can be written as

$$R = \beta(t) \frac{\pm 1 - e^{-i\pi\alpha(t)}}{\sin\pi\alpha(t)\Gamma(\alpha+1)} \nu^{\alpha(t)}. \quad (1)$$

We show that, to the same accuracy to which the Regge expansion represents a function  $F(s, t)$  for  $\nu \geq N$ , the following finite-energy sum rules should hold:

$$S_n \equiv \frac{1}{N^{n+1}} \int_0^N \nu^n \text{Im}F d\nu = \sum \frac{\beta N^\alpha}{(\alpha+n+1)\Gamma(\alpha+1)}. \quad (2)$$

The integration is defined over the right-hand cut in  $s$  and includes the Born term even if it occurs at negative  $\nu$ . We use the variable  $\nu$  rather than  $s$  because of the usual symmetry properties in  $\nu = (s-u)/2M$ . It is crucial to note that the relative importance of successive terms in the FESR (for all moments) is the same as in the usual Regge expansion, i.e., if a secondary pole or a cut is unimportant in a high-energy fit above  $N$ , then this singularity is unimportant to exactly the same extent in the FESR.<sup>1</sup>

There are several ways in which one can prove Eq. (2), the simplest given by the use of the Khuri representation.<sup>1</sup> However, it is interesting to follow the original derivation by considering superconvergence relations and their evaluation. Let us consider an antisymmetric amplitude that obeys the unsubtracted dispersion relation

$$F(\nu) = \frac{2\nu}{\pi} \int_0^\infty \frac{\text{Im}F(\nu')}{\nu'^2 - \nu^2} d\nu'. \quad (3)$$

If its leading Regge term has  $\alpha < -1$ , it will obey the superconvergence relation

$$\int_0^\infty \text{Im}F(\nu) d\nu = 0. \quad (4)$$

However, if the leading Regge term (but not the next one) is above  $-1$ , we can subtract it from  $F$ , and the resulting amplitude will obey a superconvergence relation. In order to simplify the calculations, we use instead of the conventional  $Q_{-\alpha-1}$  function of the Regge pole the simple power  $\nu^\alpha$  that has the right high-energy behavior and obeys an unsubtracted dispersion relation<sup>7</sup>:

$$R(\nu) = \frac{\beta(1 - e^{-i\pi\alpha})}{\sin\pi\alpha\Gamma(\alpha+1)} \nu^\alpha, \quad (5)$$

$$R(\nu) = \frac{2\nu}{\pi} \int_0^\infty \frac{\beta}{\Gamma(\alpha+1)} \frac{\nu'^\alpha}{\nu'^2 - \nu^2} d\nu' \quad (-1 < \alpha < 1), \quad (6)$$

The amplitude  $F - R$  will satisfy the superconvergence relation

$$\int_0^\infty \text{Im}(F - R) d\nu = 0. \quad (7)$$

Let us now derive the finite-energy sum rules. Consider a function  $F$  antisymmetric in  $\nu$  for fixed  $t$  that can be represented by a series of Regge poles for  $\nu \geq N$ . (Our derivation goes through also for the background integral in the  $j$ -plane; we leave it out for brevity of notation. Without the background integral all equations are asymptotic,  $N \rightarrow \infty$ , with the background integral they are exact for all  $N$ .) We divide the poles into three classes:  $\alpha_i$  stands for all poles which are above  $-1$ ,  $\alpha_j$  for all poles below  $-1$ , and  $\alpha_k$  for any pole that happens to be at  $-1$ . These three classes enter into the superconvergence rule in three very different ways: The poles above  $-1$  have to be subtracted from the integrand, the poles below  $-1$  do not appear at all, and the residue of the pole at  $-1$  appears on the right-hand side.

$$\int_0^\infty \left[ \text{Im}F - \sum_{\alpha_i > -1} \frac{\beta_i}{\Gamma(\alpha_i+1)} \nu^{\alpha_i} \right] d\nu = \beta_k. \quad (8)$$

Each term on the left-hand side diverges if evaluated separately. We intend to write the relation in a manifestly convergent form that will also be suitable for practical applications. Therefore, we cut off the integration at some  $\nu_{\text{max}} = N$  and express the high-energy behavior by Regge terms whose  $\alpha$  is below  $-1$ :

$$\int_0^N \left[ \text{Im}F - \sum_{\alpha_i > -1} \frac{\beta_i}{\Gamma(\alpha_i+1)} \nu^{\alpha_i} \right] d\nu + \int_N^\infty \sum_{\alpha_j < -1} \frac{\beta_j}{\Gamma(\alpha_j+1)} \nu^{\alpha_j} d\nu = \beta_k. \quad (9)$$

<sup>7</sup> The  $Q_{-\alpha-1}(\nu)$  form consists of a whole sequence of Khuri poles  $\nu^\alpha$ . If we take the expansion in terms of  $Q_{-\alpha-1}$  seriously, then the Mandelstam symmetry forces us to invoke additional Regge poles at least at every point where  $\alpha$  passes through a half integer because coefficients of the series of  $Q_{-\alpha-1}$  become infinite. Therefore, even theoretically it is more appropriate to consider a power series rather than a Legendre series.

<sup>6</sup> V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966); V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966).

The poles  $\alpha_j$  below  $-1$  have now entered the sum rule, but in a quite different manner from the poles  $\alpha_i$  above  $-1$ . We also notice that all integrals are now convergent. Performing the integration, we find the following finite-energy sum rule:

$$S(N) = \frac{1}{N} \int_0^N \text{Im} F d\nu = \sum_{\alpha_i > -1} \frac{\beta_i N^{\alpha_i}}{\Gamma(\alpha_i + 2)} + \sum_{\alpha_j < -1} \frac{\beta_j N^{\alpha_j}}{\Gamma(\alpha_j + 2)} + \beta_k N^{-1} = \sum_{\text{all } \alpha} \frac{\beta_i N^\alpha}{\Gamma(\alpha + 2)}. \quad (10)$$

There are two very important features of Eq. (10) that must be stressed: (a) All Regge terms enter in the final equation in the same form, regardless of whether  $\alpha$  happens to be above, at, or below  $-1$ . This is a big advantage of our method because it eliminates the special role that the point  $-1$  has in the usual treatment of the superconvergence relation. This latter can, of course, be rederived from (10) by letting  $N \rightarrow \infty$ , if all  $\alpha$  are below  $-1$ . (b) The relative importance of successive terms in the finite-energy sum rule (10) is the same as in the usual Regge expansion of the function  $F$ , i.e., if a secondary pole or a cut is unimportant in a high-energy fit above  $N$ , then this singularity is unimportant to exactly the same extent in the low-energy sum rule. Note that we have not used the Regge representation for  $\nu$  below  $N$ .

The generalization from Eq. (10) to the sum rules for arbitrary higher moments, Eq. (2), is straightforward. The meaning of our sum rules (2) is further elucidated if it is noted that they can also be derived in the following manner: First use the forward dispersion relation in order to compute the high-energy behavior of  $\text{Re}F$ . As an input, use the experimental data below  $N$  and the Regge fit to  $\text{Im}F$  above  $N$ . Afterwards, make a Regge fit to  $\text{Re}F$  and check the consistency of the two Regge fits. The consistency equations are identical to our FESR, Eq. (2), for all even  $n$ . The sum rules for odd  $n$  may be derived by considering the (unmeasurable)  $J$ -parity amplitude. There has been some discussion about whether fixed poles exist at the integer  $J$  values of the wrong signature. These poles would simply appear on the right-hand side of Eq. (2). Schwarz<sup>8</sup> derived superconvergence sum rules assuming that such fixed poles are absent, while Mandelstam and Wang<sup>9</sup> as well as Jones and Teplitz<sup>9</sup> recently showed that they might exist.

The finite-energy sum rules, Eq. (2), can be similarly derived for the symmetric functions, and we can write similar sum rules for negative  $n$ . For the definite  $J$ -parity amplitude  $f(\nu)$ , which has only a right-hand cut,

the formula analogous to Eq. (2) is

$$\int_0^N \frac{\text{Im} f(\nu)}{\nu^{m+1}} d\nu - \sum_i \frac{\beta_i}{\Gamma(\alpha_i + 1)} \frac{N^{\alpha_i - m}}{\alpha_i - m} = \pi \frac{f^{(m)}(0)}{m!}, \quad m \geq 0$$

$$= 0, \quad m < 0 \quad (11)$$

where  $f^{(m)}$  is the  $m$ th derivative of  $f$ .

The special case of Eq. (11) with  $m=0$  is particularly interesting. If the leading trajectory has  $\alpha \geq 0$ , then a symmetric amplitude will have to have a subtraction constant. This constant can be determined from Eq. (11) by using the Regge parameters. Alternatively, if this constant (e.g., the scattering length) is known, it helps in determining the high-energy fit. Igi<sup>10</sup> used essentially this argument in establishing the existence of the  $P'$  trajectory.

It should be noticed that for the various sum rules of arbitrary moments, it is an equally good approximation to keep only one, or several, Regge poles. For each moment, the relative contribution of the successive Regge terms is of the same order of magnitude. For illustration, consider the sum rules  $S_0$  and  $S_1$  with the following hypothetical choice of leading Regge poles—one at  $\alpha=0.5$  and the other at  $\alpha=-1.5$ . If we subtract the leading pole only, and if we use the formulation with  $N \rightarrow \infty$ , then a superconvergence relation will hold for  $S_0$  but not for  $S_1$ . On the other hand, in our treatment we see from Eq. (10) that the relative error committed by neglecting the second Regge pole is almost the same for both sum rules: The error committed in  $S_1$  is 5/3 times the error in  $S_0$ , and in both cases it is of the order  $N^{-2}$ .

Although higher-moment FESR are equally valid, their usefulness diminishes the higher the moment gets because the integrals are sensitive only to the behavior of the function immediately below  $N$ . Similarly, the high-inverse-moment FESR lose their usefulness because they determine the  $m$ th derivative of the function at threshold from the detailed behavior of the function just above threshold.

Our finite-energy sum rule  $S_0$  sheds light on the behavior of the usual superconvergence relation as the dominant Regge pole moves up through  $-1$ . Recall that the usual relation gives zero if the dominant pole is slightly below  $-1$ , a finite number (namely, the residue) if the pole is exactly at  $-1$ , and infinity if the pole is slightly above  $-1$ . Our finite relation for  $\int_0^N \text{Im} f d\nu$  always gives a term of the form  $cN^{\alpha+1}$  which is much smaller than  $c$  (but not zero) for  $\alpha$  below  $-1 - (\ln N)^{-1}$ , about equal to  $c$  for  $\alpha$  in the range  $-1 \pm (\ln N)^{-1}$ , and much larger than  $c$  (but not infinite) for  $\alpha$  above  $-1 + (\ln N)^{-1}$ . We see that the violent jumps from 0 to

<sup>8</sup> J. H. Schwarz, Princeton University Report (unpublished).

<sup>9</sup> S. Mandelstam and L. L. Wang, *Phys. Rev.* **160**, 1490 (1967); C. E. Jones and V. L. Teplitz, *ibid.* **159**, 1271 (1967).

<sup>10</sup> K. Igi, *Phys. Rev. Letters* **9**, 76 (1962).

$c$  to  $\infty$  have disappeared due to the finite integration, and that the  $J$ -plane has become "coarse-grained" with an uncertainty of  $\Delta J \sim (\ln N)^{-1}$ .

This discussion helps us to remove apparent problems that were raised recently. Muzinich<sup>11</sup> and Phillips<sup>12</sup> pointed out that the existence of a double  $\rho$  cut might spoil the  $\Delta I = 2$  superconvergence relation of the  $B$  amplitude that had been checked by many authors.<sup>13</sup> Since this cut will have  $\alpha_c(0) = 2\alpha_\rho(0) - 1$ , the exact value of  $\alpha_\rho$  seemed to be very crucial. We learn from the FESR that for practical applications it is not crucial at all whether  $\alpha_c(0)$  is slightly below or above zero. In any case, the deviations of the sum rule from zero should tell us the approximate strength of the cut or any other singularity.

The trouble with this particular example (as with many other ones) is that the only way to check the sum rule is by summing over direct channel resonances. The results have such big ambiguities that it is very difficult to draw any convincing conclusion. However, there exists at least one clean case where good experimental data is available for the amplitude, namely the  $\pi N$  charge exchange in the forward directions. Several authors<sup>1,2</sup> applied  $S_0$  to the  $A'^{(-)}$  amplitude at  $t=0$  using the known total  $\pi N$  cross section. One finds that  $S_0$ , as well as  $S_1$  and  $S_2$ , are in agreement with the conventional high-energy one  $\rho$ -pole analysis within the experimental errors.<sup>14</sup> In the present paper, we present an analysis of both  $A'^{(-)}$  and  $B^{(-)}$  amplitudes as a function of  $t$  and use the different FESR to determine their Regge parameters.

### III. APPLICATION OF THE FINITE-ENERGY SUM RULES: THE NEW BOOTSTRAP

There are several ways in which one can use the FESR. The first one is in the spirit of the original superconvergence paper of de Alfaro *et al.*,<sup>15</sup> namely, to use as input the information about the Regge terms inferred from high-energy data in order to determine parameters of the  $s$ -channel resonances.<sup>16</sup> In applying FESR to superconvergent amplitudes, one might assume that the right-hand side is negligible compared to the individual contributions from the resonances to the left-hand side of the equation. This assumption is implicitly made by all authors who saturate superconvergence relations by a finite number of resonances. This saturation amounts

<sup>11</sup> I. J. Muzinich, Phys. Rev. Letters **18**, 381 (1967).

<sup>12</sup> R. J. N. Phillips, Phys. Letters **24B**, 342 (1967).

<sup>13</sup> P. Babu, F. J. Gilman, and M. Suzuki, Phys. Letters **24B**, 65 (1967); B. Sakita and K. C. Wali, Phys. Rev. Letters **18**, 31 (1967); G. Altarelli, F. Buccella, and R. Gatto, Phys. Letters **24B**, 57 (1967).

<sup>14</sup> The sum rule  $S_1$  excludes the possibility of a strong fixed pole at  $j = -2$  in  $A'^{(-)}$ .

<sup>15</sup> V. de Alfaro, S. Fubini, G. Furlan, and G. Rossetti, Phys. Letters **21**, 576 (1966).

<sup>16</sup> L. A. P. Balázs and J. M. Cornwall, Phys. Rev. **160**, 1313 (1967). This use of the FESR is closest in spirit to the original superconvergence calculation of V. de Alfaro, S. Fubini, G. Furlan, and G. Rossetti, Phys. Letters **21**, 576 (1966).

to a choice of low  $N$  in the corresponding FESR. The FESR can, of course, be applied in the same spirit to amplitudes that do not die very fast, provided their high-energy behavior is known. In this case, the Regge parameters enter in the right-hand side and, assuming saturation by low resonances, one can determine their coupling constants. This method applies<sup>16</sup> particularly well to  $A'^{(-)}$  and  $B^{(+)}$  of  $\pi N$  scattering, since in these amplitudes the contributions of the resonances to the sum rules tend to cancel each other.

A second way<sup>1-3</sup> of using the FESR is as consistency equations that are taken together with the high-energy data for a better over-all determination of the Regge parameters. In this way we can make use of the low-energy data as well as the high-energy data in determining the asymptotic behavior. This link between the two regions which is provided by the FESR stems, of course, from the analyticity assumptions that were the basis of the derivation.

A third kind of application,<sup>3</sup> which might very well be the most interesting one from the theoretical point of view, is the use of the low-energy data alone as an input to predict the exchanged Regge poles. This is a new kind of bootstrap calculation. Thus, if one assumes a one-pole fit, one can predict its trajectory  $\alpha(t)$  by using the algebraic equation

$$S_n: S_m = (\alpha + m + 1) : (\alpha + n + 1). \quad (12)$$

The point at which  $\alpha$  passes through a right-signature integer corresponds to a pole in the scattering amplitude.

The crucial point which enables us to predict this exchanged pole is the fact that  $\text{Im}F$  (i.e., the absorptive part in the direct channel) stays regular at the position of this pole, while  $\text{Re}F$  shows the singularity<sup>17</sup>

$$\begin{aligned} \text{Im}F &\sim \nu^\alpha, \\ \text{Re}F &\sim \nu^\alpha / (t - m^2). \end{aligned} \quad (13)$$

Since  $\text{Im}F$  remains finite, there is no particular change in Eq. (12) at this point and it is easily applicable. Since we use the partial-wave series to construct  $\text{Im}F$ , we assume that the exchanged pole is much larger than the double-spectral function and neglect the latter so that the series converges. Once  $\alpha(t)$  is determined, one can use it in any of the FESR to determine  $\beta(t)$ . It is advisable to work separately with the odd and even moment sum rules, since one of these families contains the wrong-signature nonsense poles that do not affect the observable amplitude.

This bootstrap method has several advantages over the conventional  $N/D$  method: (1) Equation (12) is an algebraic condition in  $\alpha$  and the physical pole is reached when  $\alpha$  passes through an integer. This is much easier to use than the condition that the solution  $D$  of the integral equation vanishes. (2) Approximations to our method (like the resonance approximation) are

<sup>17</sup> In contrast, the absorptive part in the  $t$  channel is  $\nu^{\frac{1}{2}}(t - m^2)$ .

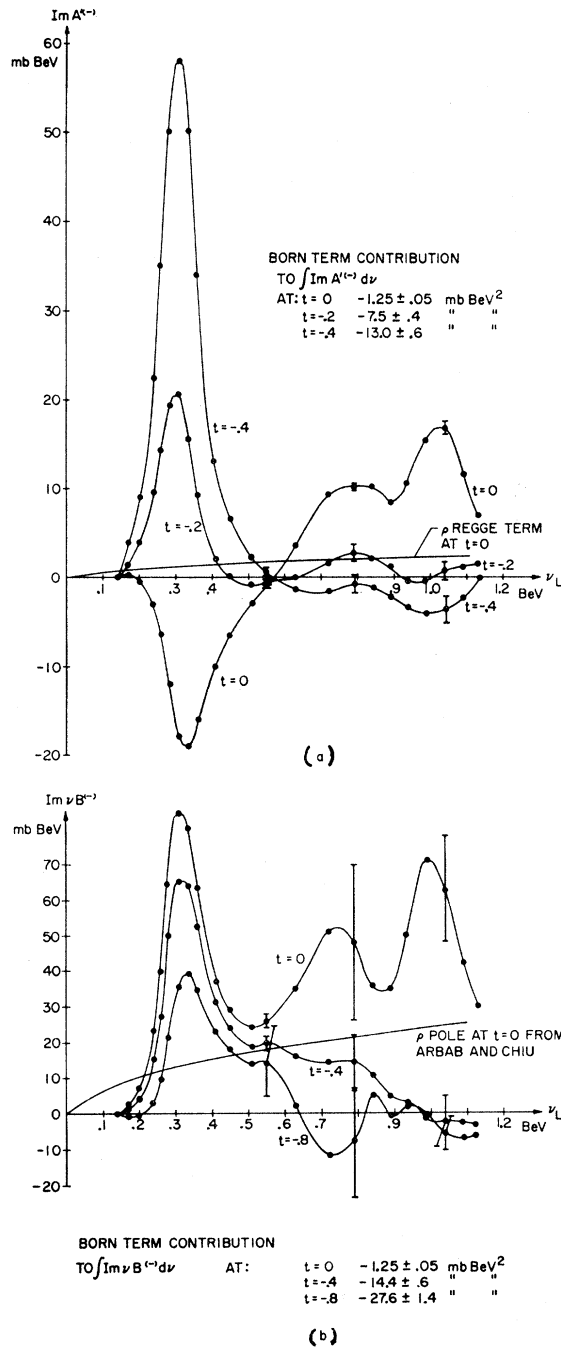


FIG. 1. The imaginary parts of  $A^{(-)}$  and  $\nu B^{(-)}$  as determined from Bareyre's phase shifts. Error bars show the variation between different phase-shift groups.

easily tested while, e.g., the determinantal approximation to  $N/D$  is not. (3) Our method uses measurable quantities, while  $N/D$  uses left-hand cuts, i.e., "forces." (4) The FESR refer to one amplitude at a time, whereas the  $N/D$  method often requires many-channel calculations. (5)  $N/D$  relies heavily on two-particle unitarity which becomes useless at intermediate energies. Our

method depends on analyticity in two complex variables,  $s$  and  $t$ .

An ultimate perfect bootstrap based on the FESR could be attempted in a self-conjugate problem (like  $\pi\pi$  scattering) in which the same particles that appear on the trajectories in the  $t$  channel are used as resonances building up the amplitudes in the low-energy  $s$  channel. We do not attempt such a program here, but rather restrict ourselves to a more practical one in which we can use experimental data, the  $\pi N$  problem. Our new type of bootstrap works particularly well for amplitudes like  $B^{(-)}$  or  $A^{(+)}$ , where all prominent resonances ( $\Delta, N_{\alpha}, N_{\gamma}$ ) enter with the same sign and add up constructively in the FESR. On the other hand, the use of the FESR in the reverse direction<sup>16</sup> (e.g., predicting the relative strength of  $N$  and  $N^*_{1238}$ ) is particularly suitable for  $A^{(-)}$  and  $B^{(+)}$ , where the resonances enter with alternating signs and tend to cancel in the FESR. These cancellations become particularly large for negative  $t$  (see below).

#### IV. DETERMINATION OF $\rho$ PARAMETERS IN $\pi N$ CHARGE EXCHANGE: USE OF A SINGLE FESR

##### A. Low-Energy Data

In applying FESR off the forward direction, we can either use phase shifts or assume saturation by the established resonances. The latter method neglects a nonresonating background; therefore we work with phase shifts. They are known only at low energies and only semiquantitatively; therefore, we expect to find only qualitative features. We have used the phase shifts of Bareyre *et al.*<sup>18</sup> to construct the imaginary parts of the amplitudes plotted in Fig. 1. The error bars indicate the variation obtained using phase shifts of different groups.<sup>19</sup> These errors turn out to be of the same order of magnitude as the nonresonating background missing in the resonance saturation model. A more quantitative treatment will be possible as soon as phase shifts are known more accurately and up to higher energies. Figure 2 compares the result of Bareyre's phase shifts with the resonance approximation for  $\text{Im} \nu B^{(-)}$  at  $t=0$ , and one can see that both methods give about the same result. Had we included error bars in this figure, both curves would have become essentially equivalent. In the following, we will rely principally on Bareyre's phase

<sup>18</sup> P. Bareyre, C. Brickman, A. V. Stirling, and G. Villet, Phys. Letters **18**, 342 (1965).

<sup>19</sup> B. H. Brandsen, P. J. O'Donnell, and R. G. Moorhouse, Phys. Letters **19**, 420 (1965); A. Donnachie, R. G. Kirsopp, A. T. Lea, and C. Lovelace, submitted to the 1966 Berkeley High-Energy Physics Conference (unpublished); we used the solutions  $A, B, H, I$ . The large error bars were usually caused by solution  $B$ . The full error bar gives the difference between the highest and lowest value so obtained. We have reason to believe that some of Lovelace's solutions may be wrong because they show an extremely strong peak in the full amplitude  $\text{Im} \nu B^{(-)}$  at  $k=1.19$  BeV/c, where no resonance seems to exist in any of the partial waves. It is therefore quite possible that the error bars in Fig. 1 are too big.

shifts with the upper limit of integration chosen at his highest point,  $\nu_L = 1.13$  BeV. However, in order to reach some additional conclusions that involve a higher cutoff, we will also make use of the known parameters of the higher resonances. Concerning the choice of  $N$  we note that in the case  $t=0$ , where one uses the well-known total cross sections, one should pick  $N$  at that energy, where the wiggles become smaller than the systematic experimental error. This occurs around  $\nu_L = 3.5$  to 4.0 BeV. In contrast the resonance model requires a lower  $N$ , because at high energies resonances alone do not saturate the amplitude. We have to find a compromise between the requirement of high  $N$  on the right-hand side of (2) and of low  $N$  on the left-hand side. We feel that a good choice is  $\nu_L = 1.5$  BeV for  $B^{(-)}$  and 2.5 BeV for  $A'^{(-)}$ . In this paper we use phase shifts, and therefore we must take the cutoff at  $\nu_L = 1.13$  BeV.

The low cutoff is less serious for  $\text{Im}\nu B^{(-)}$  than for  $\text{Im}A^{(-)}$ , because all resonances add constructively in the former, but they have alternating signs in the latter. Furthermore, the low cutoff is a better approximation for negative  $t$  than for  $t=0$  or  $t=m_\rho^2$ , because for  $t \sim -0.4$  higher resonances are suppressed, since they have  $z_s \sim 0$  and  $|P_l| < (\frac{1}{2}\pi l \sin\theta)^{-1/2}$ . The  $N$  and the  $\Delta(1238)$  blow up because their  $z_s$  is  $\ll -1$ , i.e., they are outside the physical region.

FESR at (fixed) negative  $t$  involve integrations over an unphysical region at low energies. Extrapolation beyond  $z = -1$  is done by using the partial-wave expansion. This expansion must diverge when we reach the region of the double spectral function, but even independent of that the extrapolation differs greatly from one phase-shift solution to another. (Phase-shift solutions only involve a finite number of  $l$ 's, therefore they converge everywhere.) Because of these extrapolation difficulties at large momentum transfers and low energy, one could use higher-moment sum rules which deemphasize the low-energy region.

**B. Zeros of the Regge Amplitudes**

In Fig. 1(b) we plot  $\text{Im}\nu B^{(-)}$  for various values of  $t$ . The nucleon Born term gets very large and negative

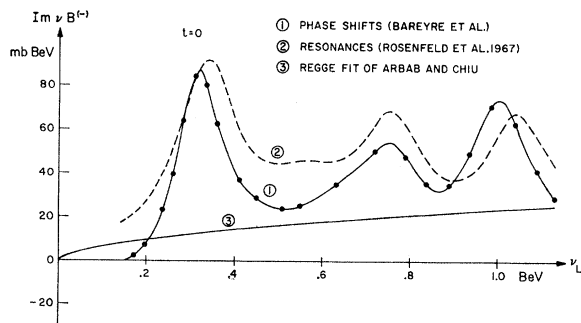


FIG. 2. Comparison between the phase-shift results, the resonance model (see Ref. 35), and the Regge term for  $\text{Im}\nu B^{(-)}$  at  $t=0$ .

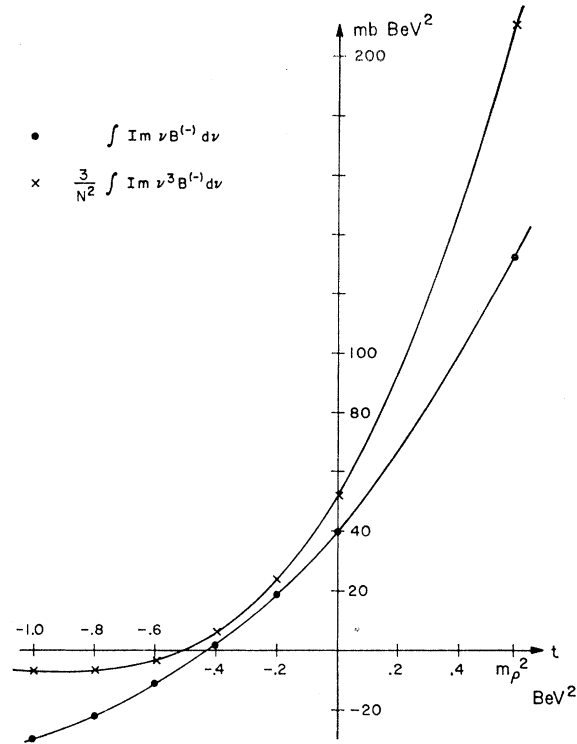


FIG. 3. Plot of the left-hand sides of FESR:  $NS_1(B^{(-)})$  and  $3NS_3(B^{(-)})$ .

for negative  $t$ , and the continuum also decreases rapidly; therefore, when evaluating the integral of this function, we find a zero between  $t = -0.4$  and  $-0.5$  BeV<sup>2</sup> (see Fig. 3). From the dominance of the Born term and the smallness of the continuum, we conclude that such a zero has to occur also if the cutoff  $N$  is chosen around 3 or 5 BeV. Thus we predict from low-energy data that the high-energy amplitude has a zero as expected from the  $\rho$  Regge-pole interpretation

$$\text{Im}\nu B^{(-)} \rightarrow d\nu^\alpha = \beta\alpha\nu^\alpha / \Gamma(\alpha+1). \tag{14}$$

In Fig. 1(a) we plot  $A'^{(-)}$  at several  $t$  values. In the integral  $S_0$  we find large cancellations, but the net result (Fig. 6) is negative for  $t \leq -0.2$ . For  $t \geq 0$ , there are serious cutoff problems, but at  $t=0$  we use  $\sigma_{\text{tot}}$  with a high  $N$  and establish that  $A'^{(-)}$  changes sign between  $t=0$  and  $t=-0.2$ . Let us discuss this problem a little more in detail. The low cutoff is a serious problem for  $\text{Im}A'^{(-)}$ , if we want to predict the Regge residue. But if we go to  $t \lesssim -0.2$ , the cutoff error due to the wiggles becomes much less important, and it becomes comparable to or smaller than the Born term error and the phase-shift error. For  $t = -0.2, -0.4, -0.6$ , the fit of Arbab and Chiu is  $2\frac{1}{2}$  to 4 times outside our error bars.<sup>20</sup> But the possibility remains that a satellite pole accounts for this difference. In order to make sure that we test the  $\rho$  pole, we choose the cutoff at  $\nu_L = 2.5$  BeV, where

<sup>20</sup> F. Arbab and C. B. Chiu, Phys. Rev. 147, 1045 (1966).

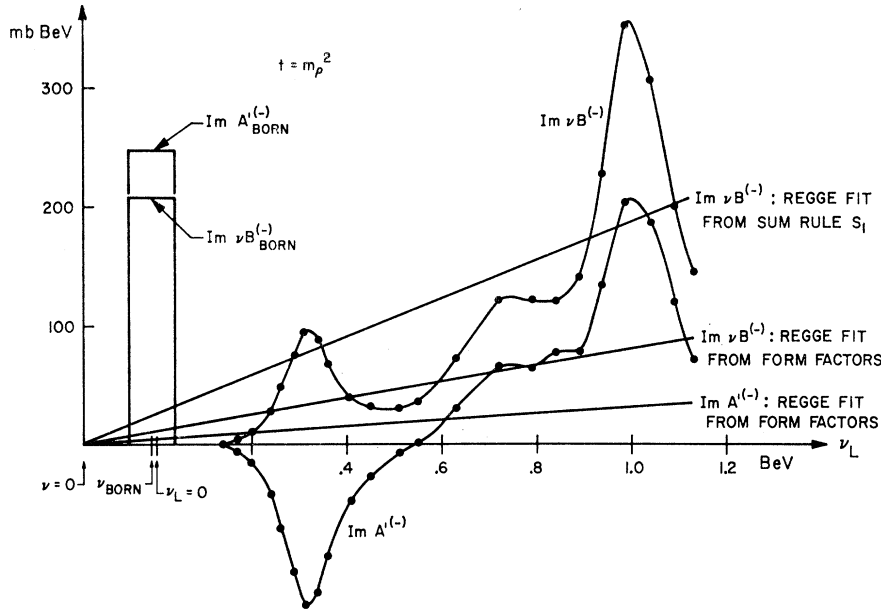


FIG. 4.  $\text{Im} \nu B^{(-)}$  and  $\text{Im} A^{(-)}$  at  $t = m_\rho^2$ .

the influence of the satellite is smaller. We fill the gap between 1 and 2.5 BeV by using bounds on  $\text{Im} A^{(-)}$  which we get from the  $\pi N$  CEX differential cross section. This bound is particularly restrictive at  $t = -0.6$ , where Carroll<sup>21</sup> observes the dip. Using the bound  $|\text{Im} A^{(-)}| \leq 2.5 \text{ mb BeV}$  between  $\nu_L = 1$  and 2.5 BeV, we get

$$c/(\alpha+1) = -2.0 \pm 1.8 \text{ mb}$$

at  $t = -0.6 \text{ BeV}^2$ . Therefore, it is established that the Regge residue function  $c$  changes sign between  $t = 0$  and  $t = -0.6$ .

Figure 4 shows  $\text{Im} \nu B^{(-)}$  and  $\text{Im} A^{(-)}$  at the interesting point  $t = m_\rho^2$ . We see that at  $t = m_\rho^2$  the higher reso-

nances become relatively more important; compare the 1688 with the 1238 in  $\text{Im} \nu B^{(-)}$ . Here we expect a straight-line Regge term and we find that the coupling of  $B^{(-)}$  seems to be bigger than expected. We will return to this question below.

Looking at  $A^{(-)}$  in Fig. 1(a), we find an interesting feature: For  $-0.4 \geq t > -1 \text{ BeV}^2$  very large cancellations take place between the Born term and the  $\Delta(1238)$ , while the higher resonances are suppressed. The reason is that at these values of  $t$ , the  $N$  and  $\Delta$  are outside the physical region (their  $z_s \ll -1$ ) while the higher resonances are inside it. This can be clearly seen from Fig. 5 which shows the relevant part of the Mandelstam plot

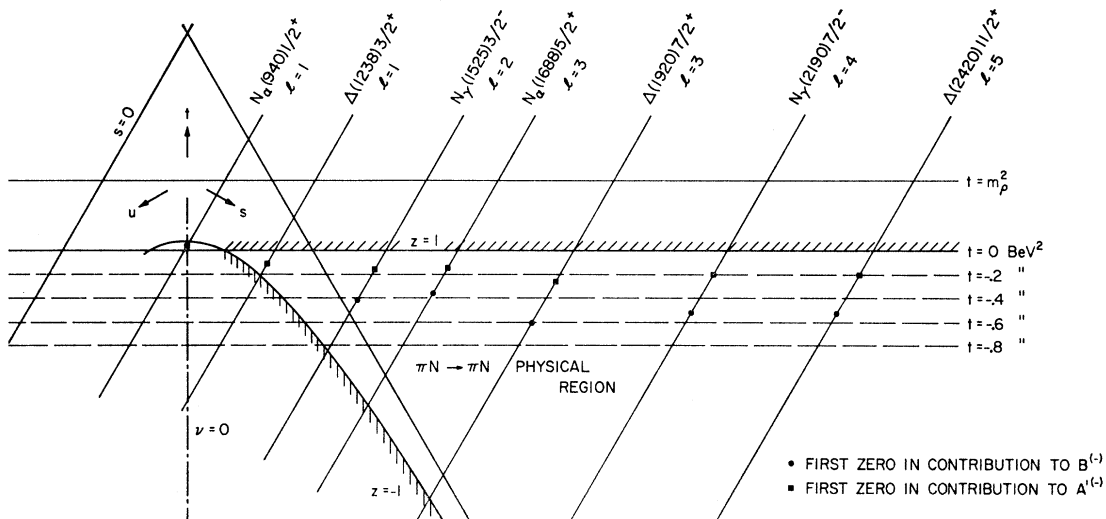
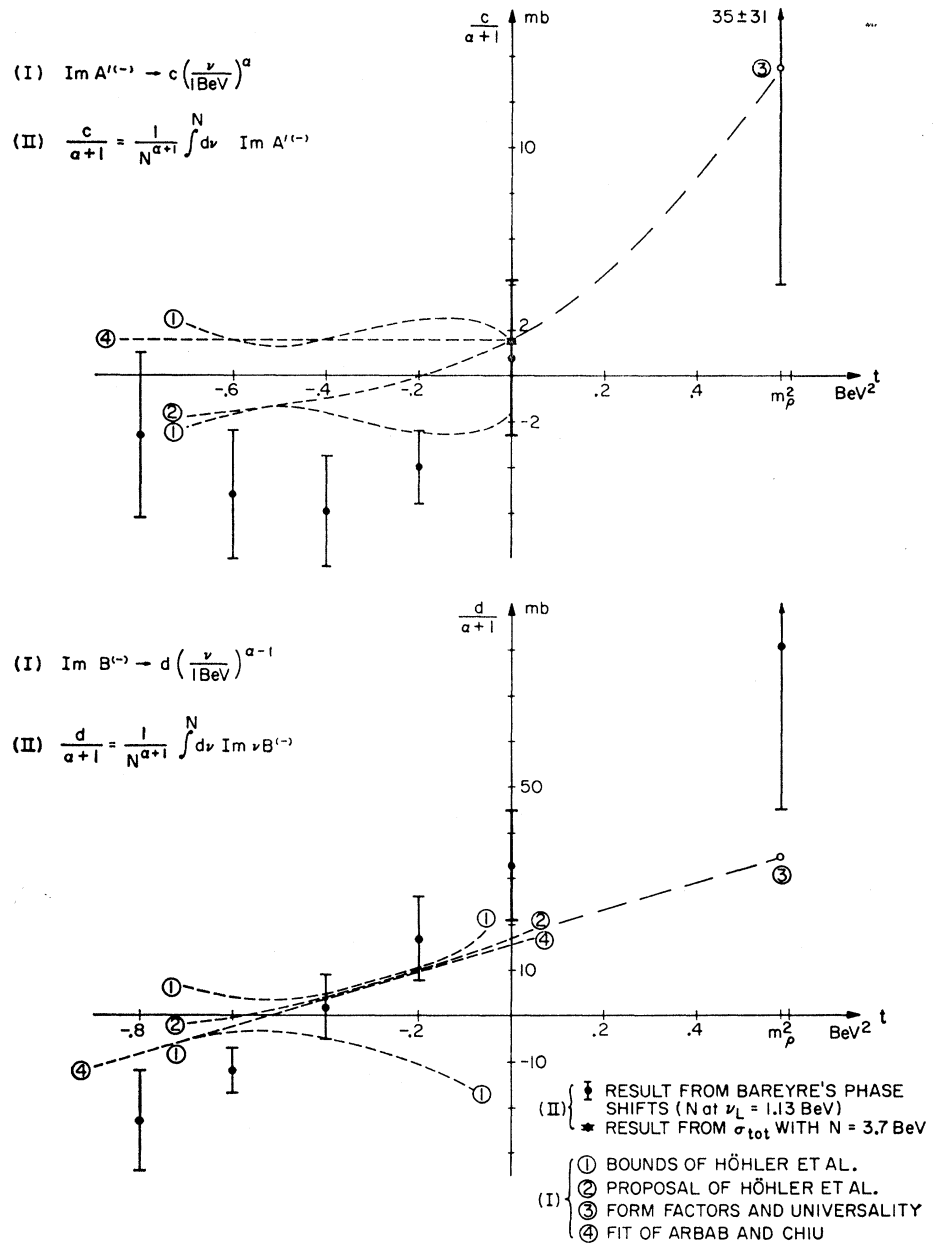


FIG. 5. First zeros of the prominent resonances on the Mandelstam plot for the  $\pi N$  problem.

<sup>21</sup> A. S. Carroll *et al.*, Phys. Rev. Letters 16, 288 (1966).





cussed by Höhler *et al.*<sup>22</sup> and is shown in Fig. 6 (see the two different fits of curves 2 and 4). Experiment gives upper bounds on  $c^2$  and  $d^2$ , shown as curve 1. The FESR allow us to resolve these ambiguities of high-energy fits: Fig. 6 shows qualitatively that the choice of 2 of Ref. 22 is preferred over choice 4 of Ref. 20, i.e.,  $c$  changes sign near  $t = -0.1$  or  $-0.2$  BeV<sup>2</sup>.

Quantitatively (see Ref. 23 for an explanation of the errors), Fig. 6 shows that the one-pole approximation using the conventional values of  $\alpha$  is outside the bound 1 established by high-energy experiments. This discrepancy (by a factor of about 2) can be fitted either by introducing a second  $\rho$  pole or by taking one "effective  $\rho$  pole," whose  $\alpha_{\text{eff}}(t)$  has to be 0.3 lower in order to give the correct predictions at 10 BeV. Choosing  $N$  small enables us to see the effect of additional singularities although it introduces big errors in the sum rules.<sup>23</sup> The higher  $N$  gets, the better the one-pole fit will be.

#### D. Intermediate $\pi N$ Resonances and the $\rho$ Regge Term

While the lowest moment sum rules  $S_1(B^{(-)})$  and  $S_0(A'^{(-)})$  connect Regge parameters to the low-energy behavior ( $N, N^*_{1233}$ ), the next higher moments  $S_3(B^{(-)})$  and  $S_2(A'^{(-)})$  establish a very interesting connection between Regge parameters and the intermediate-energy region.<sup>24</sup> The resonances in the intermediate energy region have the very suggestive features (1'), (2'), and (3') mentioned in the Introduction and shown in Fig. 5. The first zeros in  $B^{(-)}$  of the prominent, intermediate-energy resonances occur "simultaneously" in the narrow interval  $-0.6 < t < -0.4$  BeV<sup>2</sup>.<sup>25</sup> Hence,  $S_3(B^{(-)})$ , which is dominated by the intermediate-energy region will pass through zero at about the same place; see Fig. 3 and Ref. 26. We see that the  $N^*$  resonances occur with just the right quantum numbers and at the right intervals so as to vanish at the point where the Regge pole passes through zero. Similarly, the first zeros of the various resonances in the  $A'^{(-)}$  amplitude occur almost simultaneously near  $t = -0.2$  BeV<sup>2</sup>. Therefore,  $S_2(A'^{(-)})$  has a zero at about the same place. Note that  $B$  is mainly determined by  $P_l'(z)$ ,  $A'$  mainly by  $P_l(z)$ ; this

<sup>23</sup> The error bars in Fig. 6 include the following: (a) the experimental error in the low-energy integral (Born term,  $f^2 = 0.081 \pm 0.003$ , and phase shifts); and (b) an estimate of the background integral in the  $j$  plane as derived from the size of the wiggles. We estimated this error to be one-half of the area of the combined 1520–1688 peak above the smoothed-out amplitude. This error from the wiggles would rapidly diminish with a higher limit of integration  $N$ .

<sup>24</sup> The higher moments  $S_5$ , etc., cease to be useful since the integrals are dominated by the high-energy region (just below the limit of integration  $N$ ).

<sup>25</sup> If the direct-channel resonances lie on straight trajectories, then their first zeros do not stay at fixed  $t$ ; rather they approach  $t = 0$  like  $1/s$ , as can be seen from  $P_l(\cos\theta) \rightarrow J_0(l \cdot \theta)$  for  $l \rightarrow \infty$ . However, this is irrelevant in this connection, because at higher energies these resonances are unimportant due to their small elasticities.

<sup>26</sup> Taking  $\alpha$  from high-energy fits, we find good agreement between  $(\alpha+1)S_1$  and  $(\alpha+3)S_3$ , as would be expected from a one-pole model.

explains why the first zero of  $A'$  occurs at smaller momentum transfer than the first zero of  $B$ .

Thus intermediate-energy data also establish both the zero in  $B^{(-)}$  as expected from the high-energy fit (the dip phenomenon) and the zero of  $A'^{(-)}$  similar to that of curve 2 in Fig. 6.

### V. DETERMINATION OF $\rho$ PARAMETERS: COMBINED USE OF TWO FESR

#### A. One Effective $\rho$ Pole

We now consider  $B^{(-)}$  and use  $S_1$  and  $S_3$  together to determine  $\alpha_{\text{eff}}(t)$  in the bootstrap sense of Sec. III. Equation (12) reads in this case, because  $B \sim \nu^{\alpha-1}$ ,

$$\alpha = \frac{3S_3 - S_1}{S_1 - S_3}. \quad (15)$$

$3NS_3$  and  $NS_1$  are plotted in Fig. 3. We find

$t$	Prediction of $\alpha$ from Eq. (15)	Prediction of $\alpha$ from high-energy data (Refs. 20, 22, 27)
$m_\rho^2$	$1.0 \pm 0.3$	1
0 BeV <sup>2</sup>	$0.4 \pm 0.2$	0.56–0.7
-0.2 BeV <sup>2</sup>	$0.3 \pm 0.3$	0.34–0.40

Equation (15) becomes unstable when  $\alpha \rightarrow 0$  because  $S_1$  and  $S_3$  both become zero. In an "ideal" case where there is nothing but one Regge pole, the two functions would of course touch (but not cross) at the point  $\alpha = 0$ . It is therefore much more reliable to consider here the zeros of  $S_1$  and  $S_3$  directly. The zero of  $S_1$  occurs at  $t = -0.43 \pm 0.1$  BeV<sup>2</sup>, while that of  $S_3$  is at  $t = -0.52 \pm 0.1$  BeV<sup>2</sup> (see Fig. 3).

A simultaneous zero of  $S_1$  and  $S_3$  could also be caused by  $d = 0, \alpha \neq 0$ . In that case the curves  $S_1$  and  $S_3$  would cross, but this obviously is not the case in Fig. 3. We get a surprisingly good agreement, but we note that for  $-0.6 \leq t \leq 0$  the trajectory seems to be too low by 0.2. This is a second, completely independent indication that the effective  $\alpha(t)$  of a one-pole fit at  $\nu_L = 1.13$  BeV should be lower than the  $\alpha(t)$  measured around 10 BeV.

#### B. Determination of the Second $\rho$ Pole

Let us now use  $S_1$  and  $S_3$  of  $B^{(-)}$  together with the high-energy fit of Arbab and Chiu ( $\rho_1$ ) to determine the location of a possible second  $\rho$  trajectory ( $\rho_2$ ), which is an explanation of the discrepancy found in Fig. 6. Using the analog of Eq. (12),

$$\frac{S_1 - d_1 N^{\alpha_1/(\alpha_1+1)}}{S_3 - d_1 N^{\alpha_1/(\alpha_1+1)}} = \frac{\alpha_2 + 3}{\alpha_2 + 1}, \quad (16)$$

we (a) obtain an intercept  $\alpha_2(0) = 0.2 \pm 0.3$ ; (b) compute  $\alpha_2$  for  $-0.8 \leq t \leq 0$  and fit it with a straight line  $\alpha_2(t) = 0.1 + 1.2t$ ; and (c) find the zero of  $\alpha_2$  at  $t = -0.35 \pm 0.05$

<sup>27</sup> K. J. Foley *et al.*, Phys. Rev. Letters 19, 330 (1967).

from the vanishing of  $S_1 - d_1 N \alpha^1 / (\alpha_1 + 1)$ . Combining (a), (b), and (c) we get a "best fit"

$$\alpha_2(t) = 0.3 + 0.8t. \quad (17)$$

Note the large error in (a) and the difference between (b) and (c). One should be cautious and not take these values too seriously because they were arrived at by using Eq. (16). This equation is unreliable, because in it we take differences twice, first between the sum rules and the high-energy values, then between the different moments.

This second  $\rho$  pole could be the phenomenological representation of a cut; experimentally, one cannot distinguish between a cut and one or a few poles. The description by cuts is phenomenologically advantageous only when the discontinuity across the cut can be predicted from theory in terms of one or two free parameters. In general, a cut is given by a continuum of parameters, while a pole is given by only two (position, strength).

### C. Polarization

Using  $\alpha_2(t)$  as given in Eq. (17), a slightly modified Höhler-type<sup>22</sup> fit to the high-energy data (we moved the zero of  $A'^{(-)}$  from  $t = -0.2$  to  $t = -0.1$ ), and our integrals  $S_0(A'^{(-)})$  and  $S_1(B^{(-)})$ , we compute the residue functions  $c_2$  and  $d_2$  of the second  $\rho$  pole. This allows us to predict the polarization in  $\pi N$  CEX. Note that there are no free parameters left. We predict the correct sign, but the magnitude has large errors. We get  $P = +0.4 \pm 0.3$  for  $t = -0.2$  and  $k = 5.9$  BeV/c. The corresponding experimental value<sup>28</sup> is  $P = +0.15 \pm 0.20$ .

The sign of the polarization is determined by  $c_2 d_1 - c_1 d_2$  (where  $c$  and  $d$  refer to  $A'^{(-)}$  and  $B^{(-)}$ ). The sign of this determinant can be read off from Fig. 6. We see that both ratios  $c_2 : c_1$  and  $d_2 : d_1$  are positive and roughly of magnitude 1. If these ratios were exactly equal, we could not get any polarization, even in the presence of the second  $\rho$  pole. However, closer examination of Fig. 6 shows that  $c_2/c_1 > d_2/d_1$ ; therefore, we predict a positive polarization.

### D. Fixed Pole in $B^{(-)}$

An amazing result comes from evaluating  $S_0$  for  $B^{(-)}$ . The ratios between the left-hand side of the FESR and the right-hand side in the one-pole approximation (using the high-energy parameters of Ref. 20) are 4.5, 5.1, 6.1, 7.3, 8.9 for  $t = 0, -0.2, -0.4, -0.6, -0.8$ , respectively. The  $S_0$  sum rule is therefore drastically violated in the one-pole approximation, i.e., there must be a strong second pole. We determine its position by considering both sum rules  $S_0$  and  $S_1$  together with the  $\rho$  pole

parameters of Arbab and Chiu<sup>20</sup>:

$$\left( \int^N d\nu \operatorname{Im} B^{(-)} - \frac{d_1}{\alpha_1} N \alpha_1 \right) / \left( \frac{1}{N} \int^N d\nu \operatorname{Im} \nu B^{(-)} - \frac{d_1}{\alpha_1 + 1} N \alpha_1 \right) = \frac{\alpha_2 + 1}{\alpha_2}. \quad (18)$$

We get for the position of the second pole  $\alpha_2 = 0.12, 0.04, -0.03, -0.09, -0.15$ , for  $t = 0, -0.2, -0.4, -0.6, -0.8$ , respectively. In other words, it is very near  $\alpha = 0$  and has an unusually flat slope. The effect is consistent with the existence of a fixed pole at  $\alpha = 0$  that cannot affect the physical amplitude because it is at a wrong-signature nonsense point. The absence of such poles would lead to a Schwarz-type sum rule.<sup>8</sup> Mandelstam and Wang<sup>9</sup> have recently shown that fixed poles arise due to effects of the third double spectral function. Note that the pole found in our treatment is an additive one and will not change the conclusion about the dip of the  $B^{(-)}$  amplitude.

It is interesting to note the special role that the nucleon Born term plays in this sum rule. The Born term has  $\nu_L = -0.01$  BeV<sup>2</sup>. Because of the smallness of  $\nu = \nu_L + t/4M$ , it is overpowering in  $B^{(-)}$ , moderately important in  $A'^{(-)}$  and  $\nu B^{(-)}$ , and negligible in higher sum rules. Overpowering nucleon dominance in  $B^{(-)}$  means that the high-energy tail  $1/\nu$  of the Born term cannot combine with the continuum to give  $s^{\alpha-1}$  with  $\alpha \neq 0$ , therefore it gives a fixed Regge pole at  $\alpha = 0$  which is an unphysical point for  $B^{(-)}$ . Furthermore, the Born term and the 1238 enter with the same sign in  $B^{(-)}$ , but with the opposite sign in  $B^{(+)}$ .

## VI. RESONANCES, REGGE TERMS, AND DOUBLE COUNTING

### A. Interference Model and Double Counting

There are two complete representations of any scattering amplitude: One is the partial-wave series which can be dominated by direct-channel resonances or might have a large nonresonating background, and the other is the Regge asymptotic series consisting of pole (and cut) terms  $s^\alpha$  plus a background integral in the  $j$  plane. The combined FESR tells us that the sum of Regge terms  $s^\alpha$  gives a fit to the smoothed-out experimental curve, and only the remaining wiggles are contributed by the background integral in the  $j$  plane, i.e., the smoothed-out resonance contribution is already included by the Regge-pole terms. The size of these wiggles forms the limit of the accuracy to which we can determine the Regge parameters from the FESR. As such, they were included in the error attached to the calculation of the FESR.

An interference model<sup>6</sup> has been proposed which represents the amplitude as

$$F = F_{\text{Regge}} + F_{\text{Res}} \quad (19)$$

<sup>28</sup> P. Bonamy *et al.*, Phys. Letters 23, 501 (1966).

in the intermediate-energy region. In contrast, we shall show that the correct prescription should be

$$F = F_{\text{Regge}} + F_{\text{Res}} - \langle F_{\text{Res}} \rangle, \quad (20)$$

where  $\langle F_{\text{Res}} \rangle$  denotes the locally averaged resonance amplitude. The term  $\langle F_{\text{Res}} \rangle$  is not included in the interference model. If all resonances enter with the same sign, then  $\langle F_{\text{Res}} \rangle \neq 0$  and the interference model involves double counting. If the resonances enter with alternating sign and comparable strength, then  $\langle F_{\text{Res}} \rangle \simeq 0$  and the interference model agrees with our prescription.

If the resonances overlap to a considerable extent,  $F_{\text{Res}}$  becomes very smooth,  $F_{\text{Res}} \simeq \langle F_{\text{Res}} \rangle$ , and we have

$$F \simeq F_{\text{Regge}}, \quad (21)$$

even though  $F_{\text{Res}}$  might be large (compare  $\pi^+p$  at  $180^\circ$ , Sec. VID).

### B. FESR Versus Interference Model

In the interference model it is usually assumed that one Regge pole is sufficient for the high-energy region, and the resonances have to be superimposed on it in the intermediate-energy region. Let us see now how the FESR contradicts this assumption. Choose a non-flip elastic amplitude at  $t=0$  whose imaginary part is positive definite. Then let us choose  $N$  in the asymptotic region where the amplitude is given to a high accuracy as  $\nu^\alpha$ . The FESR state that below  $N$  the imaginary part has to average out<sup>29</sup> around the curve  $\nu^\alpha$ , whereas the interference model would suggest that the resonances (which necessarily are all positive) should be added to  $\nu^\alpha$  and therefore yield a curve which is everywhere above  $\nu^\alpha$ . This latter suggestion is in clear contradiction with the FESR and, therefore, with analyticity.<sup>30</sup>

Analyticity by itself does not contradict a model in which  $\nu^\alpha$  fits only a background amplitude on which one superimposes some Breit-Wigner terms, since every term satisfies the analyticity conditions separately. However, in that case the resonance terms contribute a real amplitude that goes like  $\nu^{-1}$  (the sum of their tails), which is nothing but an additional fixed Regge pole at  $\alpha=1$ . If we were to include also the latter in  $F_{\text{Regge}}$ , we would introduce double counting. In other models which suppress the high-energy tails of resonances, the first-mentioned contradiction with analyticity directly follows.

This discussion has some implications regarding recently suggested explanations of the polarization in  $\pi N$

<sup>29</sup> Because of the different weights in the different sum rules, the averaging out must occur locally and not only in the entire interval from threshold to  $N$ .

<sup>30</sup> The interference model has led R. Gatto [Phys. Rev. Letters **18**, 803 (1967)] and G. V. Dass and C. Michael [Phys. Rev. **162**, 1403 (1967)] to sum rules which, if applied to  $B^{(-)}$ , produce a violent contradiction by equating a sum of positive resonance residues to zero. While these authors use the interference model down to threshold, it must be emphasized that Barger and Cline proposed the model for intermediate energies and not for low energies.

CEX by interference of either the tails of low-energy resonances<sup>31</sup> or high-energy resonances<sup>32</sup> with the  $\rho$  Regge amplitude. The tails of the low-energy resonances go like  $\nu^{-1}$ , and if they could be isolated from the rest, they would produce a fixed Regge pole at  $\alpha=-1$ . But in fact these tails cannot be isolated, and they combine with other contributions to make up the real part of the usual  $\rho$ -Regge term (and perhaps of additional  $\rho'$ -Regge terms). Similarly, the high-energy resonances contribute their share to the sum of Regge terms.

### C. Resonance Poles and the Asymptotic Regge Expansion

A series of Regge-pole terms cannot converge for real energies below any resonance. This is due to the fact that each term  $s^\alpha$  is regular at the position of the resonance pole  $s_R = m_R^2 - im_R \Gamma_R$ . If the sum of the  $s^\alpha$  terms would converge for real  $s_c < m_R^2$ , then it would also converge everywhere outside the circle  $|s| > s_c$  and be finite at  $s = s_R$ , i.e., the amplitude would not have a pole at  $s = s_R$ . Therefore a Regge expansion is an asymptotic expansion.

A finite sum of Regge terms treats the region above the real  $s$  axis (the first sheet) and the region below it (the second sheet) alike, while we know from general principles that the scattering amplitude is regular on the first sheet, but contains all kinds of singularities on the second sheet. The situation in the background integral (for a fixed path in the  $l$  plane) is different: Again each term  $s^l$  is regular at  $s = s_R$ , but the domain of convergence is changed. The background integral converges for  $\text{Im}s > 0$  and diverges for  $\text{Im}s < 0$  (i.e., on the second sheet just below the physical region), where the resonance poles are.

It has been argued that the inclusion of more and more Regge terms could produce wiggles in analogy to the convergent expansion of  $\sin(1/s)$  in powers of  $1/s$ . This is impossible because in any asymptotic expansion we must keep fewer and fewer (Regge) terms as we go to lower energies. Wiggles could only be produced where successive terms in the asymptotic expansion are of comparable magnitude. In that case, however, the asymptotic series, by its very definition, has to be truncated before these terms are reached.

We shall now give an example which illustrates double counting, and which shows how resonance wiggles come from the background integral and not from successive Regge terms. Our model consists of the function  $F(\nu) = \psi(\nu) - \psi(1-\nu)$ , where  $\psi(\nu)$  is the logarithmic derivative of  $\Gamma(\nu)$ . The spectral function consists only of resonance poles (we do not introduce a Regge pole), and these resonances generate the full amplitude.  $F(\nu)$  has an

<sup>31</sup> R. J. N. Phillips, Nuovo Cimento **45**, 245 (1966); R. K. Logan and L. Sertorio, Phys. Rev. Letters **17**, 834 (1966).

<sup>32</sup> B. R. Desai, D. T. Gregorich, and R. Ramachandran, Phys. Rev. Letters **18**, 565 (1967); G. Altarelli, A. Borgese, F. Buccella, and M. Colocci, Nuovo Cimento **48A**, 245 (1967).

essential singularity at infinity (an accumulation point of resonance poles); however, a dispersion relation is still valid. In fact, along any ray  $\epsilon \leq \arg \nu \leq \pi - \epsilon$ ,  $F(\nu)$  allows an asymptotic expansion (Stirling's expansion) which consists of one term only:  $i\pi\nu^0$ . Two things are remarkable: (1) Although we started with resonances only, we end up with a Regge pole at  $\alpha=0$ . (2) There is only one term in the asymptotic expansion, all other terms in the Stirling expansion (an infinite number of terms) cancelled when we took the odd part of  $\psi(\nu)$ . The complex  $l$  plane contains only one Regge pole. In addition, it has an essential singularity at  $l=\infty$ .

The asymptotic formula is valid for  $|\nu| \rightarrow \infty$ ,  $\epsilon \leq \arg \nu \leq \pi - \epsilon$ . If we take  $\nu = |\nu|e^{i\epsilon}$ , then the "width" of the "resonance" poles will increase linearly, and  $F(\nu)$  will look very similar to  $\nu B^{(-)}$  in  $\pi N$  CEX at  $t=0$ , i.e., the resonances overlap more and more and form a smooth function at high energy, namely the Regge asymptotic form.

The fact that we have one Regge pole only implies that an infinite number of superconvergence relations are valid after the subtraction of this one Regge pole.

Next we treat a similar model for  $t \neq 0$ . We assign successive even angular momenta to the resonances. If we want to ensure that the first zero<sup>33</sup> of all  $P_i(z)$  occurs at the same fixed  $t$ , we must assume a tower of direct channel resonances with  $l_i \rightarrow a\sqrt{s_i}$  (or  $m_R \sim l_R$ ). The coupling constants are taken to have a power behavior  $g_i^2 \propto s_i^c$ . The asymptotic behavior of the resulting function is  $\text{Im}F(s,t) \rightarrow \beta(t)s^{\alpha(t)}$  with  $\alpha(t) = \text{const} = c - \frac{1}{2}$  and  $\beta(t) = \text{const} J_0(2a\sqrt{-t})$ .

In this simple model the  $t$ -channel Regge pole has a horizontal trajectory irrespective of the choice of the coupling constants  $g_i$ . This is no accident: One tower of direct-channel resonances is never sufficient at high energies, if we want to build up a Regge trajectory in the  $t$  channel with nonzero slope. The reason is that asymptotically for large  $l$  the Legendre polynomials have a constant ratio between the primary peak at  $z=+1$  and the secondary peak. Regge behavior demands that this ratio decreases like  $s^{\alpha(t_1)-\alpha(0)}$ , where  $t_1$  is the position of the secondary peak. A large number of partial waves (which may be represented by resonances) is required at every energy in order to suppress the secondary peaks of the various  $P_l$ . In other words, there must be a strong correlation between the partial waves, and the partial-wave description as well as the resonance saturation model becomes very uneconomical at high energies.

There is complete symmetry with respect to an interchange of  $t$  and  $u$  in our example. The  $s$ -channel resonances alone build up both a  $t$ - and a  $u$ -Regge pole. One is likely to do triple counting if one includes all three types of amplitudes (say for a fit of the valley between the forward and backward directions).

<sup>33</sup>  $P_l(\cos\theta) \rightarrow J_0(l\theta)$  for  $l \rightarrow \infty$ ; see Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., Inc., New York, 1953), Vol. II, Sec. 7.8.

## D. Interference Model at $0^\circ$ and $180^\circ$ , Crucial Tests

Crucial tests are provided by the amplitudes where all prominent resonances enter with the same sign. On the other hand, for the amplitudes where they enter with cancelling contributions, the interference model does not differ much from the correct treatment, because the double counting involves the averaged resonance contribution.

### 1. $\text{Im}k^{-1}A^{(+)}$ at $t=0$

This is the average of the  $\pi^\pm p$  total cross sections. Extrapolating the various high-energy Regge fits down to  $k=1.0$  BeV, one gets  $40 \pm 4$  mb from Foley *et al.*<sup>27</sup> and  $35 \pm 1$  mb from Rarita *et al.*<sup>34</sup> On the other hand, the locally averaged  $\sigma^{(+)}$  at 1 BeV is  $37 \pm 7$  mb, where the error bars give the size of the resonance wiggles. We see that the extrapolated  $P+P'$  Regge contribution already gives the locally averaged low-energy amplitude. There is no place left for adding in the full resonance contributions of the four resonances around the 1688 ( $F_{15}, D_{15}, S_{31}, S_{11}$ ), which amount to 26 mb.

### 2. $\text{Im}\nu B^{(-)}$ at $t=0$

In Fig. 2 we plot this amplitude both as derived from phase shifts and from the resonance model.<sup>35</sup> The region above  $\nu=0.6$  BeV has the qualitative features of the intermediate-energy region, i.e., the wiggles are smaller than the average amplitude. It is comparable to the region  $2 < \nu < 3$  BeV in  $\text{Im}A^{(-)}$  (see Fig. 7), where Barger and Olsson applied their model. Figure 2 shows that in the case of  $\text{Im}\nu B^{(-)}$  the nonresonating background amplitude is small and of the opposite sign from the extrapolated Regge amplitude. This is in contradiction to the interference model, which assumes that the nonresonating background can be approximated by the Regge amplitude.

### 3. $\pi^+ p$ Differential Cross Section at $180^\circ$

There is only one prominent trajectory here, the  $\Delta_3$  (1238, 1920, etc.). Therefore, all resonances add at  $180^\circ$ . Barger and Cline<sup>36</sup> state that a "large direct-channel contribution saturates the backward  $\pi^+ p$  elastic scattering differential cross section below 4 BeV/c. Signifi-

<sup>34</sup> W. Rarita, R. J. Riddell, C. B. Chiu, and R. J. N. Phillips, *Phys. Rev.* **165**, 1615 (1968).

<sup>35</sup> A. H. Rosenfeld *et al.*, *Rev. Mod. Phys.* **39**, 1 (1967). When we say resonance model, we take the narrow-resonance approximation, i.e., a  $\delta$  function in  $\text{Im}F$ , and merely for graphical purposes we spread the  $\delta$  function out to a symmetric Breit-Wigner shape in the amplitude under consideration. According to this definition, we have  $(\nu F)_{\text{Res}} \neq \nu(F_{\text{Res}})$ , but it has the advantage that the integral over the resonance is the same as in the narrow-resonance approximation, and therefore unique. This treatment is obviously bad for the 1238 because it is so close to threshold.

By the way, it is interesting to see in Fig. 2 that the (1670, 1670, 1688, 1700) peak has a different position in curves 1 and 2. The difference is supposed to be free of poles and therefore a function without wiggles.

<sup>36</sup> V. Barger and D. Cline, *Phys. Letters* **22**, 666 (1966).

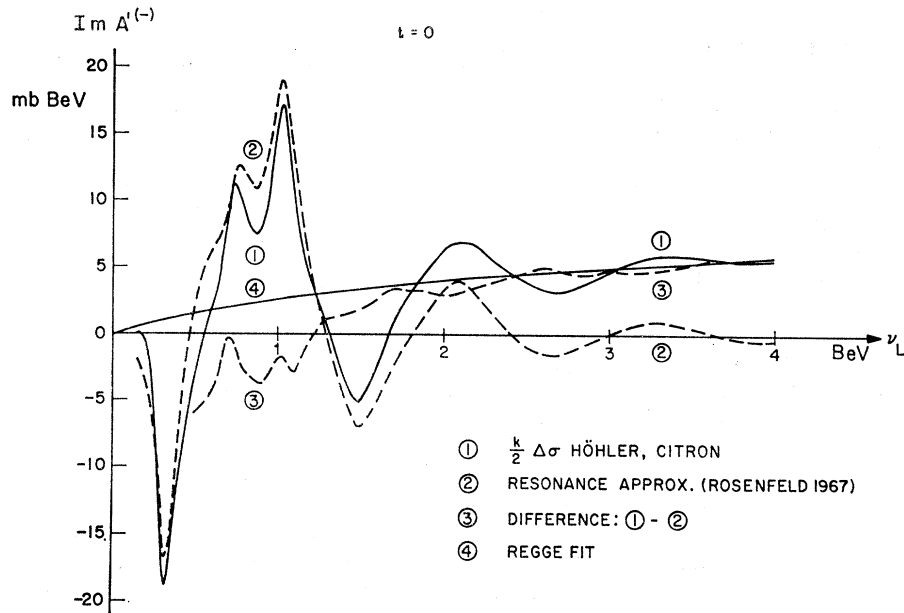


FIG. 7. Plot of  $\text{Im}A'^{(-)}$  at  $t=0$ . Comparison between different models.

cant contributions from fermion Regge exchange do not seem to be required to explain the gross features of the  $\pi^+p$  backward scattering data below 4 BeV/c."

On the other hand, using the high-energy Regge fit<sup>37</sup> down to  $p_L=3.5$  BeV (where the data and the fit of Barger and Cline are quite smooth), one also gets a good fit to the locally averaged data with Regge alone. Therefore, Regge alone and resonances alone explain the data around 4 BeV equally well. The interference model ( $F_{\text{Res}}+F_{\text{Regge}}$ ) would involve very serious double counting. This example is particularly nice, because we have not only  $F \simeq F_{\text{Regge}}$ , as is expected in general, but also  $F \simeq F_{\text{Res}}$ , i.e., we have resonance saturation in the region up to 4 BeV.

### E. Interference Model at $0^\circ$ and $180^\circ$ , Ambiguous Tests

#### 1. $\pi^-p$ Differential Cross Section at $180^\circ$

This cross section was fitted in the interference model by Barger and Cline,<sup>38</sup> apparently giving evidence for the model. However, because the resonances tend to cancel, a slight change in their parameters (elasticities) gives very different fits. In particular, Dikmen<sup>39</sup> has shown that an equally good fit can be achieved by the resonances alone if one varies the (poorly known) elasticities by a very small amount. The difference between Dikmen's and Barger's elasticities as "determined" at  $180^\circ$  is (in  $\frac{2}{3}$  of the cases) smaller than the difference between the elasticities that Barger used in his fits at  $0^\circ$  and  $180^\circ$ , respectively. On the other hand, using the high-energy Regge fit<sup>37</sup> down to  $p_L=3.5$  (where the

data are quite smooth), one also gets a good fit with Regge alone.

It is not surprising that three different models can fit the data. The two important points are as follows: (a) The resonances enter with alternating signs and the full amplitude is very small. (b) Above 3.5 BeV we have considerable overlap between resonances and therefore  $F_{\text{Res}} \simeq \langle F_{\text{Res}} \rangle$ . In Barger and Cline<sup>38</sup> the resonances cancel almost completely above 3.5 BeV,  $F_{\text{Res}} \simeq 0$ , and therefore they get essentially  $F \simeq F_{\text{Regge}}$ . In Dikmen<sup>39</sup> the resonances do not cancel completely,  $F_{\text{Res}} \neq 0$ , and because the full amplitude is small, he is able to produce  $F \simeq F_{\text{Res}}$ . In both cases the correct prescription gives  $F \simeq F_{\text{Regge}}$  because of the considerable overlap of the resonances.

#### 2. $\text{Im}A'^{(-)}$ at $t=0$

This is the difference of the  $\pi^\pm p$  total cross sections,  $\frac{1}{2}k(\sigma_- - \sigma_+)$ . This example was considered to give the second successful test of the interference model; see Barger and Olsson.<sup>6</sup> It was also the first application of FESR<sup>2</sup> which state that the full amplitude on the average equals the Regge amplitude. Figure 7 shows that this is indeed the case (with the use of only one Regge pole). Figure 7 also tells us that we have  $\langle F_{\text{Res}} \rangle \simeq 0$  because of the alternating resonance signs. Therefore, the interference model agrees with the correct prescription in this case, and it cannot be tested.

We have given conclusive experimental evidence at  $0^\circ$  and  $180^\circ$  against the interference model in all cases where the resonances enter with the same sign, and we have explained that the interference model cannot be reliably tested, if the resonances enter with alternating signs.

<sup>37</sup> A. Ashmore *et al.*, Phys. Rev. Letters **19**, 460 (1967).

<sup>38</sup> V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966).

<sup>39</sup> F. N. Dikmen, Phys. Rev. Letters **18**, 798 (1967).

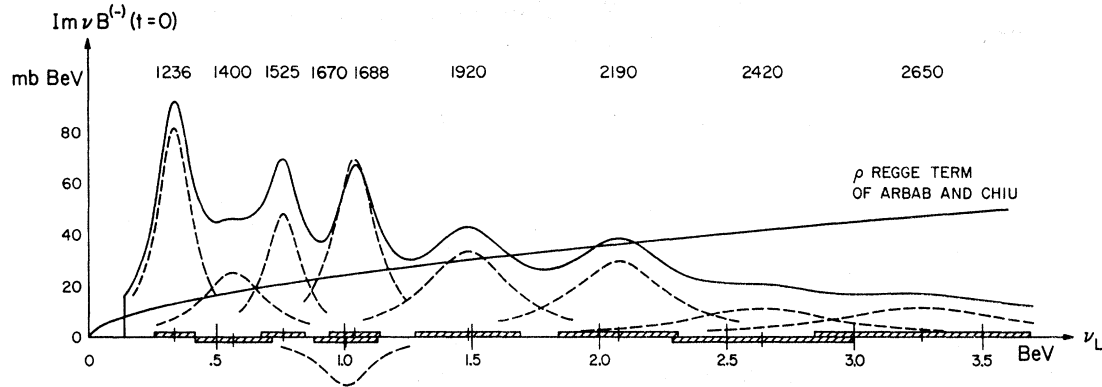


FIG. 8.  $\text{Im} \nu B^{(-)}$  at  $t=0$  as constructed in the resonance model (see Ref. 35). Dashed lines: contributions of the prominent resonances. Solid line with wiggles: sum of all resonance contributions. For comparison: Regge pole fit extrapolated to intermediate and low energies.

### F. Interference Model and Angular Distributions

Barger and Olsson<sup>6</sup> write that “inasmuch as the  $\pi^-p \rightarrow \pi^0n$  angular distribution is sharply peaked near  $0^\circ$ ,  $F_{\text{Regge}}$  is expected to contribute predominantly to the high partial waves, whereas the resonances occur in the lower partial waves.” But in fact the prominent resonances 1520, 1688, 1920, 2190, and 2420 all have a forward peak which is as sharp as the forward peak in the Regge amplitude. These resonance contributions to  $B^{(-)}$  have their first zero between  $t = -0.4$  and  $t = -0.6$ , i.e., at the same place as the Regge amplitude. Therefore,  $F_{\text{Regge}}$  and the resonances contribute predominantly to the same partial waves. Furthermore, the height of the forward peak is approximately the same for the resonances (particularly the 1920 and the 2190) and for  $F_{\text{Regge}}$ . On the other hand, the observed resonances above the 2420 are even sharper than the Regge amplitude; they have their first zero in  $B^{(-)}$  at  $t \leq -0.4$ . They belong to ultraperipheral ( $l \gg kR_0$ ) partial waves and cannot couple strongly to the elastic channel; therefore their elasticities are smaller than 10%.

Hoff<sup>40</sup> considers  $\pi^-p$  elastic scattering and shows that the dip at  $t = -0.6 \text{ BeV}^2$  and the secondary diffraction peak in the region of 2 BeV are resonance phenomena, they are explained by resonances alone, and the background amplitude  $F_B = F - F_{\text{Res}}$  has no zero. Based on this numerical result, two different conclusions have been reached: (1) Hoff identifies the background with the Regge amplitude, i.e., she uses the interference model and is therefore forced to conclude that the Regge amplitude has no zero. This contradicts high-energy measurements which show the dip. (2) In contrast, we consider the fact that  $F_{\text{Regge}}$  has a dip and a secondary peak, while  $F_B = F - F_{\text{Res}}$  shows no such structure, and we conclude that  $F_{\text{Res}}$  already includes most of  $F_{\text{Regge}}$ , i.e., we conclude that the interference model involves severe double counting in this energy region.

<sup>40</sup> G. Hoff, Phys. Rev. Letters 18, 816 (1967).

For the intermediate-energy region there are two equivalent explanations of the dip: Regge alone or resonances alone. On the other hand, for the high-energy region the explanation of the dip by resonances alone is (a) not possible with the presently known resonances; and (b) not economical in any resonance model (see the end of Sec. VIC). In the energy region to which Hoff's calculation applies, namely the region of the 2190 resonance, one finds that because the resonances add up and partly overlap, their contribution forms a fairly smooth function; e.g., in  $\text{Im} \nu B^{(-)}$ , the wiggles only amount to 7 out of 35 mb BeV, and therefore the full amplitude, the Regge amplitude, and the resonance amplitude are all approximately equal. This can be seen in Fig. 8 where we plot  $\text{Im} \nu B^{(-)}(t=0)$  in the resonance model.<sup>35</sup> For the elastic  $\pi^-p$  scattering treated by Hoff, the total cross section ( $\text{Im} k^{-1} A'_{-}$ ) is even smoother, the resonance fluctuations only amount to 1 mb out of 35 mb in that region.

In Fig. 8 we clearly see that a high cutoff is bad in the resonance model. The established resonances become a very poor approximation to the full amplitude, because their elasticities decrease exponentially. Compare Igi and Matsuda<sup>41</sup> who were the first to apply  $S_1$  to  $B^{(-)}$  at  $t=0$  using the resonance model. They obtained the ratio  $\nu B^{(-)}/A^{(-)}$  too small by a factor of 2 because of their high cutoff at  $\nu_L = 5.5 \text{ BeV}$ . In Ref. 42, Igi and Matsuda corrected this, but instead used the interference model.

### ACKNOWLEDGMENT

One of us (D. H.) would like to thank the Theory Division of the Stanford Linear Accelerator Center for their hospitality during the time when the final version of this paper was written.

<sup>41</sup> K. Igi and S. Matsuda, University of Tokyo Report (unpublished). In our Phys. Rev. Letter (Ref. 3) this reference was changed into Ref. 42 by mistake.

<sup>42</sup> K. Igi and S. Matsuda, Phys. Rev. 163, 1622 (1967).